

# SHADOWING BY NON UNIFORMLY HYPERBOLIC PERIODIC POINTS AND UNIFORM HYPERBOLICITY

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**ABSTRACT.** We prove that, under a mild condition on the hyperbolicity of its periodic points, a map  $g$  which is topologically conjugated to a hyperbolic map (respectively, an expanding map) is also a hyperbolic map (respectively, an expanding map). In particular, this result gives a partial positive answer for a question done by A. Katok, in a related context.

## 1. INTRODUCTION

Since Smale proposed the notion of *uniformly hyperbolic dynamical system*, the theory and results obtained by dynamicists around the world have described many of its features, from the structural and measure-theoretical points of view.

Nevertheless, the study of conditions for a non uniformly expanding map be expanding is not well understood, regarding the few results concerning the subject. One of these results, is the remarkable theorem of Mañé [5], valid for invariant sets without critical points for interval maps. Outside this setting, not much is known and it is by itself a interesting point of research. In particular, the study of non uniform expanding rates and conditions over a given set of points and its relations with uniform expanding behavior appears in several recent papers ([3], [9] and [10]). Let us briefly describe some of this results:

We say that a *local diffeomorphism*  $f$  is non uniformly expanding (*NUE*) on a set  $X$ , if there exists  $\eta < 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{j=0}^{n-1} \| [Df(f^j(x))]^{-1} \| \leq \eta < 0 \text{ for all } x \in X.$$

In ([3]), the authors proved that any local diffeomorphism in a compact manifold admitting non uniform expansion at a set of total probability,

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i.e., with full measure for any invariant measure, is in fact an expanding map. Similar results holds for diffeomorphisms.

By Oseledets([7]), one knows that if  $\mu$  is an invariant measure for a  $C^1$  map  $f$ , then the number

$$\lambda(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

is defined in a set of total probability and it is called *Lyapunov exponent* at  $x$  in the direction  $v$ . In [9], the author prove that if  $f$  is a local diffeomorphism such all Lyapunov exponents are positive then it is, in fact, a expanding map and also obtained the results for diffeomorphisms admitting continuous splitting (see Theorems 10 and 17).

Here, we consider the universe of systems without critical points which are topologically conjugated to expanding maps. In such context, a necessary and sufficient condition for a system to be expanding is just that it is non uniformly expanding on the set of periodic points. Therefore, as a main result, we prove that a local diffeomorphism topologically conjugated to an expanding map is itself an expanding map if, and only if, it is non uniformly expanding on the set of periodic points. We also obtain a similar result for dynamics with non uniformly hyperbolic (NUH) periodic points conjugated to an uniformly hyperbolic map.

**Theorem A.** *Let  $g : M \rightarrow M$  be a  $C^2$ -class local diffeomorphism on a compact manifold  $M$ . Suppose that  $g$  is topologically conjugated to an expanding  $C^1$  map  $f$ . If  $g$  is non uniformly expanding on the set  $Per(g)$  of periodic points, then  $g$  is an expanding map.*

**Remark 1.** We observe that the condition NUE on the periodic points is not enough to assure that the map  $g$  is expanding, even if we assume that  $g$  is topologically conjugate to an expanding map. It is a standard matter that the map  $z \rightarrow z^2$ , defined on the circle is topologically conjugated to a map with criticalities satisfying NUE condition on the periodic points. See the figure 1.

In the Theorem A, due to the fact that we are dealing with maps that are local diffeomorphisms we avoid examples as in Remark 1.

For diffeomorphisms, the existence of a continuous splitting of  $M$  plays a similar role. In [10], the authors exhibit an example of a non-hyperbolic horseshoe such that the splitting is continuous over the periodic points and all Lyapunov exponents are positive and bounded from zero. In particular, some condition of continuity of the splitting in the closure of the periodic points is necessary. In order to state our results in the invertible case, we need the following definition:

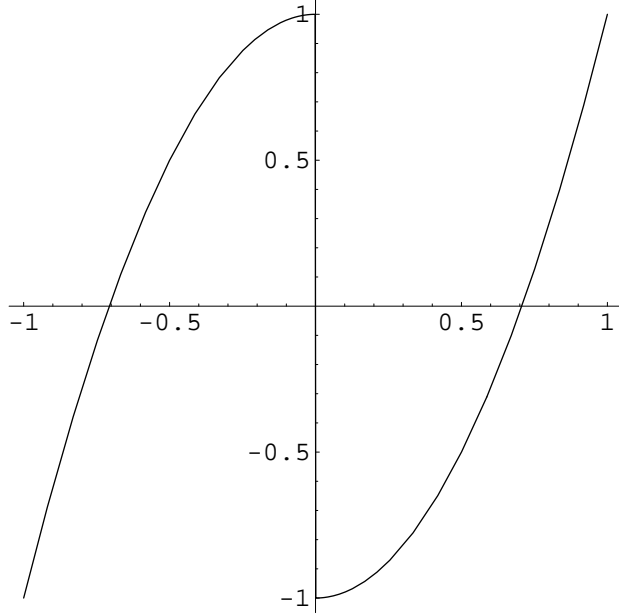


FIGURE 1. Lift to the interval of a map in  $S^1$  satisfying NUE condition on the periodic points and topologically conjugated to  $z \mapsto z^2$

**Definition 2.** (Non uniformly hyperbolic set). Let  $g : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . We say that an invariant set  $S \subset M$  is a *non uniformly hyperbolic set* or, simply, *NUH*, iff

- (1) There is an  $Dg$ -invariant splitting  $T_S M = E^{cs} \oplus E^{cu}$ ;
- (2) There exists  $\eta < 0$  and an adapted Riemannian metric for which any point  $p \in S$  satisfies

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg(g^j(p))|_{E^{cs}(g^j(p))}\| \leq \eta$$

and

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(p))|_{E^{cu}(g^j(p))}]^{-1}\| \leq \eta$$

We also recall here the notion of hyperbolic set:

**Definition 3.** Let  $\Lambda$  be an invariant set for a  $C^1$  diffeomorphism  $f$  of a manifold  $M$ . We say that  $\Lambda$  is a hyperbolic set if there is a continuous splitting  $T_\Lambda M = E^s \oplus E^u$  which is  $Tf$ -invariant ( $Tf(E^s) = E^s, Tf(E^u) = E^u$ ) and for which there are constants  $c > 0$ ,  $0 < \lambda < 1$ ,

such that

$$\|Tf^n|_{E^s}\| < c\lambda^n, \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n, \forall n \in \mathbb{N}.$$

For the diffeomorphism case, we have two slightly different results:

**Theorem B.** *Let  $g : M \rightarrow M$  be a  $C^2$  diffeomorphism on a compact manifold  $M$ , and let  $\Lambda \subset M$  be a compact invariant set. Suppose that  $g|_\Lambda$  is topologically conjugated to a  $C^1$  diffeomorphism  $f$  restricted to a set  $\hat{\Lambda}$ , hyperbolic for  $f$ . If the set  $\text{Per}(g)$  of periodic points of  $g$  is non uniformly hyperbolic (NUH), and  $T_{\text{Per}(g)}M = E^{cs} \oplus E^{cu}$  is a dominated splitting, then  $\Lambda$  is a hyperbolic set for  $g$ .*

**Theorem C.** *Let  $g : M \rightarrow M$  be a  $C^2$  diffeomorphism on a compact manifold  $M$ , and let  $\Lambda \subset M$  be a compact invariant set. Suppose that  $g|_\Lambda$  is topologically conjugated to a  $C^1$  diffeomorphism  $f$  restricted to a set  $\hat{\Lambda}$ , hyperbolic for  $f$ . If the set  $\text{Per}(g)$  of periodic points of  $g$  is non uniformly hyperbolic (NUH), and  $T_{\text{Per}(g)}M = E^{cs} \oplus E^{cu}$  has a continuous extension to a splitting on  $\overline{T_{\text{Per}(g)}M}$ , then  $\Lambda$  is a hyperbolic set for  $g$ .*

In fact, theorem B is a consequence of this last theorem C. Nevertheless, the hypotheses in B are easier to verify.

**Remark 4.** With the same proof, all results in this paper are valid if the derivative  $Dg$  is just Hölder continuous.

**Definition 5.** (Shadowing by periodic points). Let  $f : M \rightarrow M$  be a map and  $\hat{\Lambda} \subset M$  be a compact  $g$ -invariant set. We say that  $(f, \hat{\Lambda})$  has the *shadowing by periodic points property* if given  $\epsilon > 0$ , exists  $\alpha > 0$  such that for any orbit segment  $\{x, \dots, f^n(x)\} \subset \hat{\Lambda}$  with  $d(f^n(x), x) < \alpha$  there exists a periodic point  $p \in M$  with period  $n$  such that  $d(f^j(p), f^j(x)) < \epsilon$ , for all  $0 \leq j \leq n$ . In this case, we say that the orbit of  $p$   $\epsilon$ -*shadows the orbit segment*  $\{x, \dots, f^n(x)\}$ .

If  $\hat{\Lambda} \subset M$  is a hyperbolic, compact invariant set for a diffeomorphism  $f$ , then the classical theory of hiperbolic systems implies that  $(f, \hat{\Lambda})$  has the *shadowing by periodic points property* (see proposition 8.5 in [6]). The same is also valid for any system which is topologically conjugated to  $f$ . *Shadowing by periodic points* is the key ingredient in the proofs of the theorems A, B, and C we stated above in this introduction. Therefore, as a consequence of their proofs, we also obtain the following (more general) results:

**Theorem D.** *Let  $g : M \rightarrow M$  be a  $C^2$  local diffeomorphism on a compact manifold  $M$ . Suppose there exists an invariant compact set*

$\Lambda \subset M$  such that  $(g, \Lambda)$  has the shadowing by periodic points property. If  $g$  is non uniformly expanding on the set  $Per(g)$  of periodic points, then  $g$  is an expanding map on  $\Lambda$ .

**Theorem E.** Let  $g : M \rightarrow M$  be a  $C^2$  diffeomorphism on a compact manifold  $M$ , and let  $\Lambda \subset M$  be a compact  $g$ -invariant set. Suppose that  $(g, \Lambda)$  has the shadowing by periodic points property. If the set  $Per(g)$  of periodic points of  $g$  is non uniformly hyperbolic (NUH), and  $T_{Per(g)}M = E^{cs} \oplus E^{cu}$  is a dominated splitting, then  $\Lambda$  is a hyperbolic set for  $g$ .

**Theorem F.** Let  $g : M \rightarrow M$  be a  $C^2$  diffeomorphism on a compact manifold  $M$ , and let  $\Lambda \subset M$  be a compact  $g$ -invariant set. Suppose that  $(g, \Lambda)$  has the shadowing by periodic points property. If the set  $Per(g)$  of periodic points of  $g$  is non uniformly hyperbolic (NUH), and  $T_{Per(g)}M = E^{cs} \oplus E^{cu}$  has a continuous extension to a splitting on  $\overline{T_{Per(g)}M}$ , then  $\Lambda$  is a hyperbolic set for  $g$ .

## 2. THE ENDOMORPHISM CASE: NON UNIFORMLY EXPANDING PERIODIC SET

During this section,  $g : M \rightarrow M$  will always be a  $C^2$ -local diffeomorphism which is topologically conjugated to a  $C^1$  expanding endomorphism.

We recall the definition of NUE:

**Definition 6.** (Non uniformly expanding set). Let  $g : M \rightarrow M$  be a map on a compact manifold  $M$ . We say that an invariant set  $S \subset M$  is a *non uniformly expanding set* or, simply, *NUE*, iff:

There exists  $\eta < 0$  and an adapted Riemannian metric for which any point  $p \in S$  satisfies

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(p))]^{-1}\| \leq \eta$$

**Remark 7.** In this paper, we will always focus our attention on the set of periodic points  $Per(g)$  of  $g$ . Given a point  $p \in Per(g)$ , let us set  $t = t(p) := period(p)$ . In such case, the following equivalence is immediate:  $S := Per(g)$  is NUE iff there exists  $\varsigma < 1$  such that for each periodic point  $p$ ,  $\prod_{j=0}^{t(p)-1} \|[Dg(g^j(p))]^{-1}\| < \varsigma^{t(p)}$ .

For the sequence, we give a simplified definition for the case of local diffeomorphisms of the notion introduced by [4]:

**Definition 8.** (Hyperbolic time for local diffeomorphisms) Let  $z \in M$  be a regular point. We say that  $k \in \mathbb{N}$  is a  $\varsigma$ -hyperbolic time for  $z$  if for  $i = 1, \dots, k$ , holds

$$\prod_{j=1}^i \|[Dg|(g^{p-j}(z))]\|^{-1} \leq \varsigma^i. \quad (1)$$

**Lemma 9.** Suppose that  $g$  is topologically conjugated to an expanding map  $f$ . Let  $x$  be a recurrent, regular point of  $g$ . If  $\text{Per}(g)$  is NUE, then all Lyapunov exponents of  $x$  are positive.

**Proof:** Let  $\delta > 0$  such that, given any ball  $B(z, \delta)$  the corresponding inverse branches of  $g$  are well defined diffeomorphisms. Let  $\varsigma = e^\eta$ ,  $\eta$  as in definition 6,  $\varsigma < \varsigma' < 1$  fixed, and let  $\epsilon > 0$  such that  $(\sqrt{\varsigma'})^{-1} - \epsilon > 1$ . Since  $x$  is a regular point, there is  $n_0 \in \mathbb{N}$  such that

$$(\varsigma_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\varsigma_j + \epsilon)^n \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0.$$

where  $E_j$  are the Lyapunov eigenspaces and  $\log(\varsigma_j)$  are their respective Lyapunov exponents.

Now, by Pliss Lemma [8], there exists  $n_1 > n_0$  such that any point  $y$  for which we have  $\prod_{j=0}^{n-1} \|[Dg(g^j(y))]\|^{-1} \geq \varsigma^{-n}$ , for some  $n \geq n_1$ , then  $y$  has, at least,  $n_0$   $\varsigma'$ -hyperbolic times less than  $n_1$ .

We fix  $0 < \delta' \leq \delta$  such that

$$\|[Dg^{-1}(y)]\| \leq \frac{1}{\sqrt{\varsigma'}} \|Dg^{-1}(z)\|, \forall z, y; d(z, y) < \delta',$$

where  $g^{-1}$  is an inverse branch for  $g$ .

We set  $0 < \delta'' < \delta'$  such that if  $g^{-n}$  is an arbitrary composition of  $n$  inverse branches for  $g$ , then  $\text{diam}(g^{-n}(B(z, \delta''))) < \delta', \forall z \in M, \forall n \in \mathbb{N}$ . This occurs because it is valid for the hyperbolic system  $f$  to which  $g$  is conjugated.

As  $x$  is a recurrent point, we set  $n_2 \geq n_1$  a return time such that a neighborhood  $V_x \subset B(x, \delta'')$  of  $x$  is taken by  $g^{n_2}$  onto  $B(x, \delta'')$ .

Therefore, writing  $G := (g^{n_2}|_{V_x})^{-1}$ ,  $G : B(x, \delta'') \rightarrow V_x \subset B(x, \delta'')$  has a fixed point  $p \in V_x$ , which is a periodic point of period  $n_2$  for  $g$ . By hypothesis,  $p$  is a hyperbolic periodic point for which we have

$$\prod_{j=0}^{n_2-1} \|[Dg(g^j(p))]\|^{-1} \geq \|\varsigma^{-n_2}\| \Rightarrow \|DG(p)\| \leq \|\varsigma^{n_2}\|.$$

By our choice of  $n_1$  and the equation above, there exists a  $\varsigma'$ -hyperbolic time  $n_0 < n' < n_2$  for  $p$ .

Due to lemma 2.7 in [4] (see also prop. 2.23 in [2]),  $n'$  is also a  $\sqrt{\varsigma'}$ -hyperbolic time for  $x$ . In particular, this implies that

$$\|Dg^{n'}(x) \cdot v\| \geq \sqrt{\varsigma'}^{-n'} \|v\|, \forall v \in T_p M.$$

Therefore,  $\varsigma_j \geq \sqrt{\varsigma'}^{-1} - \epsilon > 1$ ,  $\forall j$ . This means that all Lyapunov exponents of  $x$  are greater than 1.  $\square$

We note that the set of Oseledet's regular, recurrent points is a total probability set, due to Oseledet's theorem and Poincaré's Recurrence theorem. This means that such set has measure equal to 1 for any  $g$ -invariant probability measure. So, for any  $g$ -invariant measure, lemma 9 implies that all Lyapunov exponents are positive. Therefore, our theorem A is obtained applying lemma 9 to the following result:

**Theorem 10.** [9] *Let  $f : M \rightarrow M$  be a  $C^1$  local diffeomorphism on a compact Riemannian manifold. If the Lyapunov exponents of every  $f$  invariant probability measure are positive, then  $g$  is uniformly expanding.*

### 3. THE DIFFEOMORPHISM CASE: NON UNIFORMLY HYPERBOLIC PERIODIC SET

Now, we treat the case when  $f$  is a diffeomorphism. Along this section, we suppose that the periodic set  $Per(g)$  is NUH. (see definition 2, on page 3).

The following remark is the analogous of remark 7 for the diffeomorphism case:

**Remark 11.** We note that the set of period points  $Per(g)$  is NUH iff there exists  $\varsigma > 1$  such that for each periodic point  $p$  with period  $t(p)$ , then  $\prod_{j=0}^{t(p)-1} \|[Dg|_{E^u}(g^j(p))]^{-1}\|^{-1} > \varsigma^{t(p)}$  and  $\prod_{j=0}^{t(p)-1} \|Dg|_{E^s}(g^j(p))\| < \varsigma^{-t(p)}$ .

Before we state and prove the next lemma let us introduce some notation. Given a periodic point  $p \in M$ , we denote the cone over  $E^s(p)$  of width  $0 < a < 1$  by

$$C_a^s(p) := \{v_s + v_u \in E^s(p) \oplus E^u(p), \text{ such that } a\|v_s\| < \|v_u\|.$$

Analogously, we define a cone over  $E^u(p)$  of width  $a$ .

Now we adapt the definition of hyperbolic times to the context of diffeomorphisms (see also [1]).

**Definition 12.** (Hyperbolic time for stable directions) Let  $0 < \lambda < 1$  and  $z \in M$  be a regular point. Suppose that  $E$  is an invariant

subbundle of  $T_{\mathbb{S}(z)}M$ , where  $\mathbb{S}(z)$  is some orbit segment of  $z$ . We say that  $k \in \mathbb{N}$  is a  $\lambda$ -hyperbolic time for  $z$  if for  $g^{-k}(z) = y$  and  $i = 1 \dots k$ , holds

$$\prod_{j=0}^{i-1} \|Dg|_E(g^j(y))\| \leq \lambda^i. \quad (2)$$

An analogous definition can be done for unstable directions just exchanging  $g$  by  $g^{-1}$  in the definition above.

**Lemma 13.** *Let  $g : M \rightarrow M$  be a  $C^2$  diffeomorphism and  $\Lambda \subset M$  be some compact  $g$ -invariant set. Suppose that  $g|_\Lambda$  is topologically conjugated to a  $f|_{\hat{\Lambda}}$ , where  $\hat{\Lambda}$  is a hyperbolic set for  $f$ . Let  $x$  be a recurrent, regular point of  $g$ . Suppose that  $\text{Per}(g)$  is NUH, and that the splitting  $T_{\text{Per}(g)} = E^{cs} \oplus E^{cu}$  have a continuous extension to  $T_{\overline{\text{Per}(g)}}M = E^1 \oplus E^2$ . Then all Lyapunov exponents of  $x$  are nonzero.*

**Proof:** Let  $\varsigma = e^\eta$ ,  $\eta < 0$  as in definition 2,  $\varsigma < \varsigma' < 1$  fixed, and let  $\epsilon > 0$  such that  $(\sqrt{\varsigma'})^{-1} - \epsilon > 1$ . Since  $x$  is a regular point, there is  $n_0 \in \mathbb{N}$  such that

$$(\varsigma_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\varsigma_j + \epsilon)^n \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0,$$

and

$$(\varsigma_j - \epsilon)^{-n} \cdot \|v_j\| > \|Dg^{-n}(x) \cdot v_j\| > (\varsigma_j + \epsilon)^{-n} \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0.$$

where  $E_j$  are the Lyapunov eigenspaces and  $\log(\varsigma_j)$  are their respective Lyapunov exponents. We denote by  $E^{cs}(x)$  (respectively,  $E^{cu}(x)$ ) the space spanned by the Lyapunov eigenspaces with negative (respectively, positive) Lyapunov exponents.  $E^0(x)$  will denote the Lyapunov eigenspace corresponding to an eventual zero Lyapunov exponent.

Let us prove that the dimension of the space  $E^{cs}(x)$  corresponding to the negative Lyapunov exponents of  $x$  is equal or greater than the dimension of the stable space of any periodic point. An analogous result will obviously hold for  $E^{cu}(x)$ . Therefore, we conclude that  $T_x M = E^{cs}(x) \oplus E^{cu}(x)$  and that all Lyapunov exponents of  $x$  are nonzero.

By taking charts, due to the uniform continuity of  $Dg$ , we can fix  $0 < a < 1$  and  $0 < \delta'$  such that if  $z$  is periodic,

$$\|Dg(y) \cdot v\| \leq \frac{1}{\sqrt{\varsigma'}} \|Dg|_{E^{cs}(z)}(z)\| \|v\|, \forall y \in B(z, \delta'), \forall v \in C_a^s(z).$$

Due to the continuity of  $E^1$ , we can assume  $a$  small enough such that each cone  $C_a^s(z)$  contains  $E^1(y)$  or all point  $y \in B(z, \delta') \cap \overline{\text{Per}(g)}$ .

Now, by Pliss Lemma in [8], there exists  $n_1 > n_0$  such that any point  $z \in \text{Per}(g)$  for which we have  $\prod_{j=0}^{n-1} \|Dg|_{E^1}(g^{-n+j}(z))\| \leq \varsigma^n$ , for some  $n \geq n_1$ , then  $z$  has, at least,  $n_0 \varsigma'$ -hyperbolic times less than  $n_1$ .

As  $x$  is a recurrent point (also for  $g^{-1}$ ), we set  $n_2 \geq n_1$  a return time for  $g^{-1}$  such that there exists one periodic point  $p$  with period  $n_2$  that  $\delta'/3$ -shadows the orbit segment  $\{x, g^{-1}(x), \dots, g^{-n_2}(x)\}$ . Such periodic point exists because  $g|_\Lambda$  is conjugated to a diffeomorphism  $f|_{\text{hat}\Lambda}$  which is shadowed by periodic points (see proposition 8.5 in [6]).

By hypothesis,  $p$  is a hyperbolic periodic point for which we have

$$\left\| \prod_{j=0}^{n_2-1} Dg|_{E^{cs}(g^j(p))} \right\| \leq \|\varsigma^{n_2}\|$$

By our choice of  $n_1$  and the equation above, there exists a  $\varsigma'$ -hyperbolic time  $n_0 < n' < n_2$  for  $p$ .

Due to prop. 2.23 in [2],  $n'$  is also a  $\sqrt{\varsigma'}$ -hyperbolic time for  $x$ . More precisely, this means that

$$\prod_{j=0}^{n'-1} \|Dg|_{E^1(g^{-n'+j}(x))}(g^j(g^{-n'}(x)))\| \leq \sqrt{\varsigma'}^{n'},$$

since the space  $E^1(g^{-n'+j}(x)) \subset C_a^s(g^{-n'+j}(p))$ .

In particular, this implies that

$$\|Dg^{-n'}(x) \cdot v\| \geq \sqrt{\varsigma'}^{-n'} \|v\|, \forall v \in E^1(x) \quad (3)$$

This implies that the dimension of the negative Lyapunov exponents space  $E^{cs}(x)$  is at least the dimension of  $E^1(x)$  which equals the dimension of  $E^{cs}(p)$ . In fact, if we had  $\dim(E^1(x)) > \dim(E^{cs}(x))$ , then since  $T_x M = E^{cs}(x) \oplus E^0(x) \oplus E^{cu}(x)$ , the intersection  $E^1(x) \cap (E^0(x) \oplus E^{cu}(x))$  would be nontrivial. That is an absurd, because no vector in  $(E^0(x) \oplus E^{cu}(x)) \setminus \{0\}$  satisfies equation 3.

Applying the same arguments above to  $E^{cu}(x)$  we conclude that the number of positive Lyapunov at  $x$  is at least the dimension of  $E^{cu}(p)$  and this concludes the lemma.  $\square$

For the next results we recall here the definition of *dominated splitting*:

**Definition 14.** (*Dominated splitting*). Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$  and let  $X \subset M$  be an invariant subset. We say that a splitting  $T_X M = E \oplus \hat{E}$  is a *dominated splitting* iff:

- (1) The splitting is invariant by  $Df$ , which means that  $Df(E(x)) = E(f(x))$  and  $Df(\hat{E}(x)) = \hat{E}(f(x))$ .

(2) There exist  $0 < \lambda < 1$  and some  $l \in \mathbb{N}$  such that for all  $x \in X$

$$\sup_{v \in E, \|v\|=1} \{\|df^l(x)v\|\} \cdot \left( \inf_{v \in \hat{E}, \|v\|=1} \{\|df^l(x)v\|\} \right)^{-1} \leq \lambda.$$

A priori, we do not require dominated splitting to be continuous. However, they always are:

**Lemma 15.** *Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . Let  $X \subset M$  be some  $f$ -invariant set. Suppose there exists some invariant dominated splitting  $T_X M = E \oplus \hat{E}$ . Then, such splitting is continuous in  $T_X M$ , and unique since we fix the dimensions of  $E, \hat{E}$ . Moreover, it extends uniquely and continuously to a splitting of  $T_{\overline{X}} M$ .*

**Proof:** By replacing  $f$  by an iterate, there is no loss of generality in supposing that  $l \in \mathbb{N}$  in definition 14 equals to 1. We start by constructing an invariant dominated splitting on  $T_{\overline{X}} M$  extending the one we have on  $T_X M$ . Let  $\mathbb{O}(x)$  be an orbit contained in  $\overline{X}$ . Our construction will be dependent of some choices. We choose one representative of  $\mathbb{O}(x)$ , for example  $x$ . Let us also choose some  $(x_n)$ ,  $x_n \in X$ ,  $x_n \rightarrow x \in M$ , as  $n \rightarrow \infty$ . Let  $v_n^1, \dots, v_n^s \in E(x_n)$ ,  $\hat{v}_n^{s+1}, \dots, \hat{v}_n^m \in \hat{E}(x_n)$  be orthonormal bases of  $E(x_n)$ ,  $\hat{E}(x_n)$ , respectively. The domination property is equivalent to

$$\|Df(x_n) \sum_{j=1}^s \alpha_j v_n^j\| \cdot \|Df(x_n) \sum_{i=s+1}^m \beta_i \hat{v}_n^i\|^{-1} \leq \lambda < 1,$$

for any convex combination  $\sum_{j=1}^s \alpha_j v_n^j$ ,  $\sum_{i=s+1}^m \beta_i \hat{v}_n^i$ . Replacing by some convergent subsequence, if necessary, we can suppose that  $(v^1, \dots, v^s)$ ,  $v^1, \dots, v^s \in T_x M$  (resp.  $(\hat{v}^{s+1}, \dots, \hat{v}^m)$ ) is the limit of the sequence  $(v_n^1, \dots, v_n^s)$  (resp. of the sequence  $(\hat{v}_n^{s+1}, \dots, \hat{v}_n^m)$ ). Since the domination property is a closed condition,

$$\|Df(x) \sum_{j=1}^s \alpha_j v^j\| \cdot \|Df(x) \sum_{i=s+1}^m \beta_i \hat{v}^i\|^{-1} \leq \lambda < 1,$$

holds.

Now, we write  $G$  for the Gram-Schmidt operator (which takes a linearly independent set of vectors on an orthonormal set of vectors spanning the same vector space). Given any iterate  $y = f^k(x)$ ,  $k \in \mathbb{Z}$ , then  $f^k(x_n) \rightarrow y$  and

$$\begin{aligned} G \circ (Df^k(x_n)v_n^1, \dots, Df^k(x_n)v_n^s) &\rightarrow G \circ (Df^k(x)v^1, \dots, Df^k(x)v^s), \\ G \circ (Df^k(x_n)\hat{v}_n^{s+1}, \dots, Df^k(x_n)\hat{v}_n^m) &\rightarrow G \circ (Df^k(x)\hat{v}^{s+1}, \dots, Df^k(x)\hat{v}^m), \end{aligned}$$

as  $n \rightarrow \infty$ . Writing  $(w_k^1, \dots, w_k^s) := G \circ (Df^k(x)v^1, \dots, Df^k(x)v^s)$  and  $(\hat{w}_k^{s+1}, \dots, \hat{w}_k^m) := G \circ (Df^k(x)\hat{v}^{s+1}, \dots, Df^k(x)\hat{v}^m)$ ,  $k \in \mathbb{Z}$ , the same calculations above show that

$$T_y M = \text{span}\{w_k^1, \dots, w_k^s\} \oplus \text{span}\{\hat{w}_k^{s+1}, \dots, \hat{w}_k^m\} =: E(y) \oplus \hat{E}(y)$$

is a dominated splitting.

Moreover, it is clear that

$$Df(f^k(x))(\text{span}\{w_k^1, \dots, w_k^s\}) = \text{span}\{w_{k+1}^1, \dots, w_{k+1}^s\}$$

and

$$Df(f^k(x))(\text{span}\{\hat{w}_k^{s+1}, \dots, \hat{w}_k^m\}) = \text{span}\{\hat{w}_{k+1}^{s+1}, \dots, \hat{w}_{k+1}^m\},$$

which implies that it is an invariant splitting.

Note that since the dominated splitting condition is a closed condition, if we prove that there exists a unique dominated splitting with the same dimensions of the splitting we constructed, it will be automatically continuous. This is because, given  $x_n \rightarrow x \in X$ , any convergent sequences of orthonormal bases of  $E(x_n), \hat{E}(x_n)$  will converge to orthonormal bases of dominated spaces in  $T_x M$  which due the uniqueness will necessarily be  $E(x), \hat{E}(x)$ .

The argument to prove uniqueness is the following. Suppose that we have two invariant dominated splittings  $T_{\overline{X}} M = E \oplus \hat{E}$ ,  $T_{\overline{X}} M = E' \oplus \hat{E}'$ . Fix an arbitrary  $x \in \overline{X}$ .

By changing  $f$  for some positive iterate  $f^l$ , there is no loss of generality in supposing that domination condition is valid for  $l = 1$  on both splittings. The domination condition yields:

$$\|df|_{E(x)}\| \left( \inf_{v \in \hat{E}, \|v\|=1} \{\|Df(x)v\|\} \right)^{-1} \leq \lambda$$

and

$$\|df|_{E(x)}\| \left( \inf_{v \in \hat{E}', \|v\|=1} \{\|Df(x)v\|\} \right)^{-1} \leq \lambda.$$

Let us show that  $E(x) = E'(x)$ . Note that if  $E(x) \subset E'(x)$  (or vice-versa), as the spaces have the same dimension, they should be the same. So, let us suppose by contradiction that there exist  $v \in E(x) \setminus E'(x)$  and  $v' \in E'(x) \setminus E(x)$ . We then write  $v = v_{E'} + v_{\hat{E}'}$ , with  $v_{E'} \in E'$ ,  $v_{\hat{E}'} \in \hat{E}'$  and  $v_{\hat{E}'} \neq 0$ . This last inequality, together with the invariance of the splittings implies that

$$Df^n(x) \cdot v = \alpha_n v_{E'}^n + \beta_n v_{\hat{E}'}^n,$$

where  $v_{E'}^n$  and  $v_{\hat{E}'}^n$  are unitary vectors respectively in  $E'(f^n(x))$  and  $\hat{E}'(f^n(x))$  and  $\alpha_n/\beta_n \lesssim \lambda^n \rightarrow 0$ . In particular,  $Df^n(x) \cdot v \in E(f^n(x))$

belongs in an arbitrarily small width cone over  $\hat{E}'(f^n(x))$  (which dominates  $E'(f^n(x))$ ), as we take  $n$  sufficiently big. This implies that, for all  $n \in \mathbb{N}$  sufficiently big, there exists  $v_n = Df^n(x) \cdot v / \|Df^n(x) \cdot v\| \in E(f^n(x))$  such that

$$\|df|_{E'(y_n)}\| \cdot \|Df(y_n)v_n\|^{-1} < \tilde{\lambda} < 1,$$

where  $y_n = f^n(x)$ . Now, we repeat the same argument above for  $v' \in E'(x) \setminus E(x)$  and again for all  $n$  sufficiently big, we obtain unitary vectors  $v'_n \in E'(y_n)$  such that

$$\|df|_{E(y_n)}\| \cdot \|Df(y_n)v'_n\|^{-1} < \tilde{\lambda} < 1.$$

Therefore, we have

$$\|df|_{E'(y_n)}\| \leq \tilde{\lambda} \cdot \|Df(y_n)v_n\| \leq \tilde{\lambda} \cdot \|df|_{E(y_n)}\|$$

and

$$\|df|_{E(y_n)}\| \leq \tilde{\lambda} \cdot \|Df(y_n)v'_n\| \leq \tilde{\lambda} \cdot \|df|_{E(y_n)}\|,$$

which is a contradiction.  $\square$

**Lemma 16.** *Suppose that  $g$  is topologically conjugated to a hyperbolic map  $f$ . Let  $x$  be a recurrent, regular point of  $g$ . Suppose that  $\text{Per}(g)$  is NUH, and that the splitting  $T_{\text{Per}(g)}M = E^{cs} \oplus E^{cu}$  is a dominated splitting. Then all Lyapunov exponents of  $x$  are nonzero.*

**Proof:** The proof is a direct consequence of lemmas 13 and 15. By lemma 15, the invariant dominated splitting over  $T_{\text{Per}(g)}M$  extends to a unique continuous invariant (dominated) splitting over  $\overline{T}_{\text{Per}(g)}M$ . So, we become under the hypotheses of lemma 13, which allows to conclude that all Lyapunov exponents of any recurrent point  $x \in M$  are nonzero.  $\square$

By the same arguments as in the expanding case (see paragraph below the proof of lemma 9), our theorem B is obtained applying lemma 13 to the following result:

**Theorem 17.** [9] *Let  $f : M \rightarrow M$  be a  $C^1$  diffeomorphism on a compact Riemannian manifold, with a positively invariant set  $\Lambda$  for which the tangent bundle has a continuous splitting  $T_\Lambda M = E^{cs} \oplus E^{cu}$ . If  $f$  has positive Lyapunov exponents in the  $E^{cu}$  direction and negative Lyapunov exponents in the  $E^{cs}$  direction on a set of total probability, then  $f$  is uniformly hyperbolic.*

## 4. ON A CONJECTURE OF A. KATOK

A. Katok has conjectured that a  $C^{1+}$  system which is Hölder conjugated to an expanding map (respectively, an Anosov diffeomorphism) is also expanding (respectively, is also an Anosov diffeomorphism).

Note that, under the hypotheses of such conjecture, the periodic points of the  $g : M \rightarrow M$  are hyperbolic, with uniform bounds for the eigenvalues of iterate of  $Dg$  in the period of such points. This is proven below.

First, we consider the expanding case. Let  $p$  a periodic point of period  $t$  of  $g$ . Then,  $h(p)$  is a periodic point of period  $t$  of  $f$ . Let us call  $f^{-1}$  the inverse branch of  $f$ , defined on a neighborhood of the orbit of  $h(p)$ , for which  $h(p) = \hat{p}$  is a periodic point of period  $t$ . Analogously, let us call  $g^{-1}$  be the inverse branch of  $g$  for which  $p$  is a periodic point of period  $t$ . Since  $f$  is an expanding map, there are  $0 < \hat{\lambda} < 1$  and  $\hat{\delta} > 0$  such that

$$d(f^{-j}(\hat{x}), f^{-j}(\hat{y})) \leq \hat{\lambda}^j d(\hat{x}, \hat{y}), \forall j \in \mathbb{N}, \forall \hat{x}, \hat{y} \in B(\hat{p}, \hat{\delta}).$$

As an immediate consequence of the  $C^\alpha$  conjugation  $h$  there exists  $\delta > 0$  such that

$$d(g^{-j}(x), g^{-j}(y)) \leq (\hat{\lambda}^\alpha)^j K^{1+\alpha} d(x, y)^{(\alpha^2)}, \forall j \in \mathbb{N}, \forall x, y \in B(p, \delta)$$

and

$$d(g^{-j}(x), g^{-j}(y)) \leq (\hat{\lambda}^\alpha)^j K^{1+\alpha} \delta^{\alpha^2}, \forall j \in \mathbb{N}, \forall x, y \in B(p, \delta). \quad (4)$$

**Proposition 18.** *Let  $B(x_0, r) \subset M$  and  $G : \overline{B(x_0, r)} \rightarrow B(x_0, r)$  a class  $C^1$  local diffeomorphism such that  $G(x_0) = x_0$  and for some  $0 < \lambda < 1$  and  $0 < \beta < 1$*

$$d(G^m(x), G^m(y)) \leq \lambda^n d(x, y)^\beta, \forall x, y \in B(x_0, r).$$

*Then all eigenvalues of  $DG(x_0)$  are equal or less than  $\lambda$ .*

**Proof:** Using charts, there is no loss of generality in supposing that  $M$  is an euclidean space and  $x_0 = 0$ . By contradiction, suppose there exists an invariant splitting  $\mathbb{R}^m = E^s + E^c$ , an adapted norm  $\|x\| = \|(x_s, x_c)\| = \max\{\|x_s\|, \|x_c\|\}$  and  $\sigma > \lambda$  such that

$$\|DG(0) \cdot x_s\| \leq \lambda \cdot \|x_s\|, \forall x_s \in E^s,$$

$$\|DG(0) \cdot x_c\| \geq \sigma \cdot \|x_c\|, \forall x_c \in E^c.$$

Let  $\epsilon > 0$  such that  $\lambda + \epsilon < \sigma - \epsilon$  and take  $\theta = \frac{\lambda + \epsilon}{\sigma - \epsilon}$ .

Therefore, there is  $\tilde{r} \leq r$  such that if we write

$$G(x) = DG(0) \cdot x + \rho(x),$$

then  $\|\rho(x)\| < \epsilon \|x\|$ ,  $\forall x$ ,  $\|x\| < \tilde{r}$ .

We define a central cone

$$V_c := \{(x_s, x_c); \|x_s\| \leq \theta \|x_c\|\}$$

By the hypothesis, there exists  $\tilde{r} \leq \tilde{r}$  such that  $G^n(B(0, \tilde{r})) \subset B(0, \tilde{r})$ ,  $\forall n \in \mathbb{N}$ . So, let us iterate  $x \in B(0, \tilde{r}) \cap V_c$  (we write  $x^n = G^n(x)$ ). We obtain:

$$\|x_c^1\| \geq \sigma \|x_c^0\| - \epsilon \|x^0\| \geq (\sigma - \epsilon) \|x_c^0\|$$

and

$$\|x_s^1\| \leq \lambda \|x_s^0\| + \epsilon \|x^0\| \leq (\lambda + \epsilon) \|x_c^0\|.$$

This implies that

$$\|x_s^1\| \leq \frac{\lambda + \epsilon}{\sigma - \epsilon} \|x_c^1\|.$$

In particular, if  $x \in B(0, \tilde{r}) \cap V_c$  then  $G(x) \in V_c$ .

Therefore proceeding inductively, we obtain

$$\|x^n\| = \|x_c^n\| \geq (\sigma - \epsilon)^n \|x_c^0\| = (\sigma - \epsilon)^n \|x^0\|.$$

This contradicts the hypothesis, which implies that

$$\|x^n\| \leq \text{const} \cdot \lambda^n, \forall n \in \mathbb{N}.$$

As  $\epsilon > 0$  is arbitrary, we conclude that any eigenvalue of  $DG(0)$  is less than  $\lambda$ . □

The proposition above implies our assertive that, if a map  $g$  is Hölder conjugated to an expanding map (respectively, Anosov) then all periodic points have only nonzero Lyapunov exponents, and such exponents are uniformly bounded away from zero. However, up to now we do not know if, for example, the mild uniformity given by a Hölder conjugation, plus the conjugation itself, imply that  $Per(g)$  is NUE.

Nevertheless, as a direct consequence of the last section, we obtain that such conjecture is valid in the case that  $Per(g)$  is NUE (respectively, for Anosov, if  $Per(g)$  is NUH with dominated splitting).

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